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MIXED-TYPE FINITE ELEMENT FORMULATION OF HIGHER ORDER SHEAR DEFORMATION THEORY FOR THE LINEAR AND NONLINEAR ANALYSES OF A LAMINATED COMPOSITE PLATE

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For the linear and nonlinear analyses of a laminated composite plate structure, the mixed type finite element program is developed on the basis of higher order shear deformation theory of laminated plates. The accuracy of this program is checked by means of comparing with the existing results for laminated rectangular plates and is found to agree well with them. Deformations and interlaminar stresses of laminated plates are calculated according to the variation of layer numbers, fiber orientations, and plate thicknesses, so that the shear and nonlinear effects on their behaviors are studied. It is found that plate deformations are reduced by means of arranging the fiber direction into the angle-ply and increasing layer numbers.

Key Words: Mixed-Type Finite Element Method, Higher order Shear Deformation Theory, Laminated Plate, Shear Effect, Nonlinear Effect, Angle-ply, Cross-ply

NOMENCLATURE

a, b : Side lengthes of a rectangular plate
E_1, E_2 : Elastic constants in fiber and matrix directions respectively
G_{12}, G_{23}, G_{31} : Shear moduli in each direction
h : Plate thickness
n : Number of sublaminates
$q = q_o\, f(x,y)$: Load intensity function
$q^* = q_o(a/h)^4/E_2$: Nondimensionalized load intensity
w^* : Dimensionless deflection
$\boldsymbol{N}$: $[N_1\ N_2\ N_3]^T = [N_x\ N_y\ N_{xy}]^T$
$\boldsymbol{M}$: $[M_1\ M_2\ M_3]^T = [M_x\ M_y\ M_{xy}]^T$
$\boldsymbol{P}$: $[P_1\ P_2\ P_3]^T = [P_x\ P_y\ P_{xy}]^T$
$\boldsymbol{V}$: $[V_1\ V_2\ V_3]^T = [V_x\ V_y\ V_{xy}]^T$
$\boldsymbol{R}$: $[R_1\ R_2\ R_3]^T = [R_x\ R_y\ R_{xy}]^T$
υ_{12} : Poisson's ratio to fiber direction in matrix
Ω : Analytical domain
$\sigma^*_x, \sigma^*_y, \tau^*_{xy}, \tau^*_{yz}, \tau^*_{zx}$: Dimensionless stress components
$\boldsymbol{\varepsilon}^0_L$: $[\varepsilon^L_1\ \varepsilon^L_2\ \varepsilon^L_3]^T = \left[\frac{\partial u_0}{\partial x}\ \frac{\partial v_0}{\partial y}\ \frac{\partial u_0}{\partial y} + \frac{\partial v_0}{\partial x}\right]^T$
$\boldsymbol{\varepsilon}^0_N$: $[\varepsilon^N_1\ \varepsilon^N_2\ \varepsilon^N_3]^T = \left[\frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2 \frac{1}{2}\left(\frac{\partial w}{\partial y}\right)^2 \frac{\partial w}{\partial x}\frac{\partial w}{\partial y}\right]^T$
$\bar{\boldsymbol{\varepsilon}}^0$: $[\bar{\varepsilon}^0_1\ \bar{\varepsilon}^0_2]^T = [\gamma^0_{yz}\gamma^0_{xz}]^T$

1. INTRODUCTION

In recent years, composite materials have been widely used, largely because of their superior mechanical properties. However, transverse shear deformations are much more pronounced for composite structures than for conventional materials. Various theories have threfore been established to analyze composite plates.

The classical theory of plates (Reissner, Stavsky, 1961) based on the Kirchhoff-Love hypothesis underestimates deflections and stresses in the laminated plate. These resulting errors are even higher for plates made of advanced composites like graphite-epoxy, whose elastic modulus to shear modulus ratio is very high.

Many plate theories exist that account for transverse shear strains. of these, the theories based on assumed displacement fields as functions of thickness coordinate provide comparatively easy applications.

In Reissener-Mindlin type theories [Reissner (1961), Stavsky (1961), Mindlin (1958), Whitney (1970, 1973), Lo (1977)], the displacement field accounts for linear or higher order variations of midplane displacements through thickness.

But these shear deformation theories do not satisfy the conditions of zero transverse shear stresses on the top and the bottom surfaces of a plate, and require a shear correction factor to the transverse shear stiffness.

Levinson (1980), Murthy (1981), and Reddy (1984) presented theories that account for a parabolic variation of the transverse shear strains through thickness, and consequently, there is no need to use the shear correction coefficients in computing the shear stresses. Displacement fields in these higher order shear deformation theories are mathematically derived from virtual work principle to be useful for actual problems.

Laminated plate being thin, its geometrically nonlinear behavior can not be neglected. For these nonlinear probelms. Chia (1972) applied the perturbation method to the nonlinear analysis of a clamped anisotropic plate but no pertinent results were reported.

Zaghloul (1975), thereafter, reported the results of the linear and nonlinear analyses of symmetrically laminated plates. Using the cassical theory of laminated plates, they obtained the nonlinear solution of simply supported laminated plates by means of finite difference technique and compared the results with experimentally measured values.

Finite element analysis of a laminated composite plate has been reported by many researchers, but most of them [Jeya-

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chandrabose (1985), Chaudhuri (1987)] are based on the classical theory of laminated plates.

For higher order shear deformation theory, Kant (1982) and Phan (1985) suggested the finite element formulation method. but they treated only the linear analysis of laminated plates, and because of using the displacement type finite element of high order shape functions, the algebraically complicated procedure of element stiffness formulation and the introduction of more degrees of freedom were necessiated and required more storages and computer times.

In this paper, for the linear and nonlinear analyses of laminated composite plates, higher order shear deformation theory of laminated plate bending is simply formulated by the mixed type finite element method.

The variation effects of layer number, fiber orientation and plate thickness on the behavior of a laminated rectangular plate are studied. The shear and nonlinear effects of deformation are characteristically evaluated.

2. HIGHER ORDER SHEAR DEFORMATION THEORY OF LAMINATED COMPOSITE PLATES

In the higher order shear deformation theory of laminated plate bending, in-plane displacements (u,v) are represented as high order function of thickness coordinate, z,on the condition which shear stress components τ_{yz}, τ_{xz} are zero on the top and the bottom surfaces of a plate. In this paper, higher order shear deformation theory of laminated plates suggested Reddy (1984) is reviewed and the following displacement field is introduced ;

$$\begin{aligned} u &= u_0 + z\left[\varphi_x - \frac{4}{3}\left(\frac{z}{h}\right)^2\left(\varphi_r + \frac{\partial w}{\partial x}\right)\right] \\ v &= v_0 + z\left[\varphi_y - \frac{4}{3}\left(\frac{z}{h}\right)^2\left(\varphi_y + \frac{\partial w}{\partial y}\right)\right] \\ w &= w(x,y) \end{aligned} \tag{1}$$

where u_0, v_0 and w denote the displacements of a point (x,y) on the midplane, and φ_x, φ_y are the rotations of normals to midplane about the y and x axes, respectively. This displacemant field is the same as that chosen by Levinson (1980) except for u_0, v_0. The von Karman strains associated with the displacement field in Eq. (1) are ;

$$\begin{aligned} \boldsymbol{\varepsilon} &= \boldsymbol{\varepsilon}^0 + z(\boldsymbol{k}^0 + z^2\boldsymbol{k}^2) \\ \bar{\boldsymbol{\varepsilon}} &= \bar{\boldsymbol{\varepsilon}}^0 + z^2\bar{\boldsymbol{k}}^2 \end{aligned} \tag{2}$$

where

$$\boldsymbol{\varepsilon}^0 = \begin{Bmatrix} \varepsilon^0{}_x \\ \varepsilon^0{}_y \\ \gamma^0{}_{xy} \end{Bmatrix} = \begin{Bmatrix} \frac{\partial u_0}{\partial x} + \frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2 \\ \frac{\partial v_0}{\partial y} + \frac{1}{2}\left(\frac{\partial w}{\partial y}\right)^2 \\ \frac{\partial u_0}{\partial y} + \frac{\partial v_0}{\partial x} + \frac{\partial w}{\partial x} \cdot \frac{\partial w}{\partial y} \end{Bmatrix} \tag{3a}$$

$$\boldsymbol{k}^0 = \begin{Bmatrix} \boldsymbol{k}^0{}_x \\ \boldsymbol{k}^0{}_y \\ \boldsymbol{k}^0{}_{xy} \end{Bmatrix} = \begin{Bmatrix} \frac{\partial \varphi_x}{\partial x} \\ \frac{\partial \varphi_y}{\partial y} \\ \frac{\partial \varphi_x}{\partial y} + \frac{\partial \varphi_y}{\partial x} \end{Bmatrix} \tag{3b}$$

$$\boldsymbol{k}^2 = \begin{Bmatrix} \boldsymbol{k}^2{}_x \\ \boldsymbol{k}^2{}_y \\ \boldsymbol{k}^2{}_{xy} \end{Bmatrix} = \begin{Bmatrix} \frac{\partial \varphi_x}{\partial x} + \left(\frac{\partial^2 w}{\partial x^2}\right) \\ \frac{\partial \varphi_y}{\partial y} + \left(\frac{\partial^2 w}{\partial y^2}\right) \\ \frac{\partial \varphi_x}{\partial y} + \frac{\partial \varphi_y}{\partial x} + 2\frac{\partial^2 w}{\partial x \partial y} \end{Bmatrix} \tag{3c}$$

$$\bar{\boldsymbol{\varepsilon}}^0 = \begin{Bmatrix} \gamma^0{}_{yz} \\ \gamma^0{}_{xz} \end{Bmatrix} = \begin{Bmatrix} \varphi_y + \frac{\partial w}{\partial y} \\ \varphi_x + \frac{\partial w}{\partial x} \end{Bmatrix} \tag{4a}$$

$$\bar{\boldsymbol{k}}^2 = \begin{Bmatrix} \boldsymbol{k}^2{}_{yz} \\ \boldsymbol{k}^2{}_{xz} \end{Bmatrix} = -\frac{4}{h^2}\begin{Bmatrix} \varphi_y + \frac{\partial w}{\partial y} \\ \varphi_x + \frac{\partial w}{\partial x} \end{Bmatrix} \tag{4b}$$

For the plate possessing a plane of elastic symmetry parallel to the x-y plane, the constitutive euqations for a layer can be written ;

$$\begin{aligned} \boldsymbol{\sigma} &= \boldsymbol{Q}\boldsymbol{\varepsilon} \\ \bar{\boldsymbol{\sigma}} &= \bar{\boldsymbol{Q}}\bar{\boldsymbol{\varepsilon}} \end{aligned} \tag{5}$$

where $\boldsymbol{Q}, \bar{\boldsymbol{Q}}$are the transformed reduced stiffness matrices (Reddy, 1984).

Using this displacement field, the equilibrium equations of a system can be derived by principle of virtual displacements as follows ;

$$\begin{aligned} \delta\pi &= \int_\Omega\left[\int_{-h/2}^{h/2}(\boldsymbol{\sigma}^T\delta\boldsymbol{\varepsilon} + \bar{\boldsymbol{\sigma}}^T\delta\bar{\boldsymbol{\varepsilon}})dz - q\delta w\right]dxdy \\ &\int_\Omega\left[\boldsymbol{N}^T\delta\boldsymbol{\varepsilon}^0 + \boldsymbol{M}^T\delta\boldsymbol{x}^0 + \boldsymbol{P}^T\delta\boldsymbol{x}^2 + \boldsymbol{V}^T\delta\bar{\boldsymbol{\varepsilon}}^0 \right. \\ &\left. + \boldsymbol{R}^T\delta\bar{\boldsymbol{x}}^2 - q\delta w\right]d\Omega = 0 \end{aligned} \tag{6}$$

where

$$(\boldsymbol{N},\boldsymbol{M},\boldsymbol{P}) = \int_{-h/2}^{h/2}\boldsymbol{\sigma}(1,z,z^3)dz \tag{7}$$

$$(\boldsymbol{V},\boldsymbol{R}) = \int_{-h/2}^{h/2}\bar{\boldsymbol{\sigma}}(1,z^2)dz \tag{8}$$

Considering (7), (8) and substituting (2), (3), into Eq. (6), the following equilibrium equtions and pertinent boundary conditions are obtained by integration by part.

$$\frac{\partial N_x}{\partial x} + \frac{\partial N_{xy}}{\partial y} = 0 \tag{9a}$$

$$\frac{\partial N_{xy}}{\partial x} + \frac{\partial N_y}{\partial y} = 0 \tag{9b}$$

$$\begin{aligned} &\frac{\partial V_x}{\partial x} + \frac{\partial V_y}{\partial y} + q - \frac{4}{h^2}\cdot\left(\frac{\partial R_x}{\partial x} + \frac{\partial R_y}{\partial y}\right) \\ &+ \frac{4}{3h^2}\cdot\left(\frac{\partial^2 P_x}{\partial x^2} + 2\frac{\partial^2 P_{xy}}{\partial x \partial y} + \frac{\partial^2 P_y}{\partial y^2}\right) \\ &+ \frac{\partial}{\partial x}\left(N_x\frac{\partial w}{\partial x} + N_{xy}\frac{\partial w}{\partial y}\right) \\ &+ \frac{\partial}{\partial y}\left(N_{xy}\frac{\partial w}{\partial x} + N_y\frac{\partial w}{\partial y}\right) = 0 \end{aligned} \tag{9c}$$

$$\frac{\partial M_x}{\partial x} + \frac{\partial M_{xy}}{\partial y} - V_x + \frac{4}{h^2}R_x - \frac{4}{3h^2}\left(\frac{\partial P_x}{\partial x} + \frac{\partial P_{xy}}{\partial y}\right) = 0 \tag{9d}$$

$$\frac{\partial M_{xy}}{\partial x} + \frac{\partial M_y}{\partial y} - V_y + \frac{4}{h^2}R_y - \frac{4}{3h^2}\left(\frac{\partial P_{xy}}{\partial x} + \frac{\partial P_y}{\partial y}\right) = 0 \tag{9e}$$

$$\bar{N}_x = N_x n_x + N_{xy} n_y = 0 \text{ or } u_o = 0 \tag{10a}$$

$$\bar{N}_y = N_{xy} n_x + N_y n_y = 0 \text{ or } v_o = 0 \tag{10b}$$

$$\bar{P}_x = P_x n_x + P_{xy} n_y = 0 \text{ or } \varphi_x + \frac{\partial w}{\partial x} = 0 \tag{10c}$$

$$\bar{P}_y = P_{xy}n_x + P_y n_y = 0 \text{ or } \varphi_y + \frac{\partial w}{\partial y} = 0 \quad (10d)$$
$$\bar{M}_x = M_x n_x + M_{xy} n_y = 0 \text{ or } \varphi_x = 0 \quad (10e)$$
$$\bar{M}_y = M_{xy} n_x + M_y n_y = 0 \text{ or } \varphi_y = 0 \quad (10f)$$
$$\bar{N}_x \frac{\partial w}{\partial x} + \bar{N}_y \frac{\partial w}{\partial y} + \frac{4}{3h^2}\left(\frac{\partial P_x}{\partial x} + \frac{\partial P_x}{\partial y}\right) + \left(V_n - \frac{4}{h^2}R_n\right) = 0 \text{ or } w = 0 \quad (10g)$$

In Eq. (10), n_x, n_y are the direction cosines of outward normal on boundary surfaces of the laminates, and $V_n = V_x n_x + V_y n_y$, $R_n = R_x n_x + R_y n_y$. Boundary conditions designated in Eq. (10) are useful for finite element analysis.

3. LINEAR AND NONLINEAR FINITE ELEMENT METHODS

From Eq. (6)

$$\delta\pi = \int_\Omega \left[\delta\boldsymbol{\varepsilon}^{0T}\boldsymbol{N} + \delta\boldsymbol{x}^{0T}\boldsymbol{M} + \delta\boldsymbol{x}^{2T}\boldsymbol{P} + \delta\bar{\boldsymbol{\varepsilon}}^{0T}\boldsymbol{V} + \delta\bar{\boldsymbol{x}}^{2T}\boldsymbol{R} - q\delta w \right] dxdy = 0 \quad (11)$$

can be written and considering (5), (7), and (8), the following matrix equations are obtained ;

$$\begin{Bmatrix} \boldsymbol{N} \\ \boldsymbol{M} \\ \boldsymbol{P} \end{Bmatrix} = \begin{bmatrix} \boldsymbol{A} & \boldsymbol{B} & \boldsymbol{E} \\ \boldsymbol{B} & \boldsymbol{D} & \boldsymbol{F} \\ \boldsymbol{E} & \boldsymbol{F} & \boldsymbol{H} \end{bmatrix} \begin{Bmatrix} \boldsymbol{\varepsilon}^0 \\ \boldsymbol{x}^0 \\ \boldsymbol{x}^2 \end{Bmatrix} \quad (12)$$

$$\begin{Bmatrix} \boldsymbol{V} \\ \boldsymbol{R} \end{Bmatrix} = \begin{bmatrix} \bar{\boldsymbol{A}} & \bar{\boldsymbol{D}} \\ \bar{\boldsymbol{D}} & \bar{\boldsymbol{F}} \end{bmatrix} \begin{Bmatrix} \bar{\boldsymbol{\varepsilon}}^0 \\ \bar{\boldsymbol{x}}^2 \end{Bmatrix} \quad (13)$$

where

$$(\boldsymbol{A},\boldsymbol{B},\boldsymbol{D},\boldsymbol{E},\boldsymbol{F},\boldsymbol{H}) = \int_{-h/2}^{h/2} \boldsymbol{Q}(1,z,z^2,z^3,z^4,z^6)dz \quad (14)$$

$$(\bar{\boldsymbol{A}},\bar{\boldsymbol{D}},\bar{\boldsymbol{F}}) = \int_{-h/2}^{h/2} \bar{\boldsymbol{Q}}(1,z^2,z^4)dz \quad (15)$$

Considering (3), (12) and (13), Eq. (11) is the general form of a displaceement type finite element method and includes several high order derivatives of displacements that require a quintic polynomial to represent the interpolation functions-which is computationally and algebraically complicated.

To compensate these problems, in this paper, we resort to the mixed-type finite element formulation with $u, v, w, \varphi_x, \varphi_y, M_x, M_y, M_{xy}, P_x, P_y$ and P_{xy} as primary variables. This mixed type finite element method requires transforming Eq. (12) to the following :

$$\begin{Bmatrix} \boldsymbol{N} \\ \boldsymbol{x}^0 \\ \boldsymbol{x}^2 \end{Bmatrix} = \begin{bmatrix} \boldsymbol{G}' & \boldsymbol{H}' & \boldsymbol{L}' \\ -\boldsymbol{H}'^T & \boldsymbol{B}' & \boldsymbol{C}' \\ -\boldsymbol{L}'^T & \boldsymbol{C}'^T & \boldsymbol{F}' \end{bmatrix} \begin{Bmatrix} \boldsymbol{\varepsilon}^0 \\ \boldsymbol{M} \\ \boldsymbol{P} \end{Bmatrix} \quad (16)$$

$$\begin{bmatrix} \boldsymbol{B}' & \boldsymbol{C}' \\ \boldsymbol{C}'^T & \boldsymbol{F}' \end{bmatrix} = \begin{bmatrix} \boldsymbol{D} & \boldsymbol{F} \\ \boldsymbol{F} & \boldsymbol{H} \end{bmatrix}^{-1}$$

$$\boldsymbol{H}' = \boldsymbol{B}\boldsymbol{B}' + \boldsymbol{E}\boldsymbol{C}'^T, \boldsymbol{L}' = \boldsymbol{B}\boldsymbol{C}' + \boldsymbol{E}\boldsymbol{F}' \quad (17)$$
$$\boldsymbol{G}' = \boldsymbol{A} - \boldsymbol{H}'\boldsymbol{B} - \boldsymbol{L}'\boldsymbol{E}$$

From (12), (13) and (16), Eq. (11) can be written as follows ;

$$\delta\pi = \int_\Omega [\delta\varepsilon^{oT} \ \delta M^T \ \delta P^T \ \delta \boldsymbol{x}^{oT} \ \delta \boldsymbol{x}^{2T}] \times \begin{bmatrix} \boldsymbol{G}' & \boldsymbol{H}' & \boldsymbol{L}' & 0 & 0 \\ \boldsymbol{H}'^T & -\boldsymbol{B}' & -\boldsymbol{C}' & 1 & 0 \\ \boldsymbol{L}'^T & -\boldsymbol{C}'^T & -\boldsymbol{F}' & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{bmatrix} \begin{Bmatrix} \boldsymbol{\varepsilon} \\ \boldsymbol{M} \\ \boldsymbol{P} \\ \boldsymbol{x} \\ \boldsymbol{x}^2 \end{Bmatrix} + [\delta\bar{\boldsymbol{\varepsilon}}^{oT} \ \delta\boldsymbol{x}^{2T}] \begin{bmatrix} \bar{\boldsymbol{A}} & \bar{\boldsymbol{D}} \\ \bar{\boldsymbol{D}} & \bar{\boldsymbol{F}} \end{bmatrix} \begin{Bmatrix} \bar{\boldsymbol{\varepsilon}}^0 \\ \bar{\boldsymbol{x}}^2 \end{Bmatrix} - q\delta w \Big] dxdy = 0 \quad (18)$$

Since nonlinear terms of Eq. (18) are included in $\boldsymbol{\varepsilon}^0$, this strain vector can be separated into two parts ;

$$\boldsymbol{\varepsilon}^0 = \boldsymbol{\varepsilon}_L^0 + \boldsymbol{\varepsilon}_N^0 = \begin{Bmatrix} \dfrac{\partial u_0}{\partial x} \\ \dfrac{\partial v_0}{\partial y} \\ \dfrac{\partial u_0}{\partial y} + \dfrac{\partial v_0}{\partial x} \end{Bmatrix} + \begin{Bmatrix} \dfrac{1}{2}\left(\dfrac{\partial w}{\partial x}\right)^2 \\ \dfrac{1}{2}\left(\dfrac{\partial w}{\partial y}\right)^2 \\ \dfrac{\partial w}{\partial x} \cdot \dfrac{\partial w}{\partial y} \end{Bmatrix} \quad (19)$$

Substituting (19) into (18), Eq. (18) can be written as follows ;

$$\delta\pi = \delta\pi_L + \delta\pi_N \quad (20)$$

$\delta\pi_L$ is the basic equation to calculate the linear stiffness matrix and can be represented as follows ;

$$\begin{aligned} \delta\pi_L = \int_\Omega \Big\{ & (g'_{ij}\varepsilon^L_j + h'_{ij}M_j + \ell'_{ij}P_j)\delta\varepsilon^L_i \\ & + \left[\bar{a}_{mm} - \frac{8}{h^2}\bar{d}_{mm} + \left(\frac{4}{h^2}\right)\bar{f}_{mm} \right] \bar{\varepsilon}^L_m \delta\bar{\varepsilon}^L_n \\ & + (h'_{ji}\varepsilon^L_j - b'_{ji}M_j - C'_{ij}P_j + \boldsymbol{x}^o_i)\delta M_i \\ & + \left(\ell'_{ij}\varepsilon^L_j - f'_{ij}P_j - C'_{ji}M_j - \frac{4}{3h^2}\boldsymbol{x}^o_i\right)\delta P_i \\ & + \left(M_i - \frac{4}{3h^2}P_i\right)\delta\boldsymbol{x}^o_i + \frac{4}{3h^2}\left[\left(\frac{\partial P_1}{\partial x} + \frac{\partial P_3}{\partial y}\right)\delta\left(\frac{\partial w}{\partial x}\right)\right. \\ & + \frac{\partial w}{\partial x}\delta\left(\frac{\partial P_1}{\partial x} + \frac{\partial P_3}{\partial y}\right) + \frac{\partial w}{\partial y}\delta\left(\frac{\partial P_2}{\partial y} + \frac{\partial P_3}{\partial x}\right) \\ & + \left(\frac{\partial P_2}{\partial y} + \frac{\partial P_3}{\partial x}\right)\delta\left(\frac{\partial w_x}{\partial y}\right) - q\delta w \Big\} dxdy \end{aligned} \quad (21)$$

$\delta\pi_N$ is the geometrically nonlinear term and the nonlinear stiffness matrix is calculated from this equation as follows ;

$$\begin{aligned} \delta\pi_N = \frac{1}{2}\int_\Omega \Big[& \Big\{ \Big[g'_{1j}\left(\frac{\partial w}{\partial x}\right) + g'_{3j}\left(\frac{\partial w}{\partial y}\right) \Big] \frac{\partial w}{\partial x} \\ & + \Big[g'_{2j}\Big[\frac{\partial w}{\partial y}\Big] + g'_{3j}\Big[\frac{\partial w}{\partial x}\Big] \Big] \frac{\partial w}{\partial y} \Big\} \delta\varepsilon^L_j \\ & + \Big\{ [g'_{1j}\varepsilon^N_j + g'_{11}\varepsilon^N_1 + 2g'_{33}\varepsilon^N_2 + g'_{1j}\varepsilon^L_j \\ & + h'_{1j}M_j + \ell'_{3j}P_j] \frac{\partial w}{\partial y} \\ & + \Big[2g'_{3j}\varepsilon^N_j + \left(\frac{1}{2}g'_{12} - g'_{33}\right)\varepsilon^N_3 \\ & + g'_{3j}\varepsilon^L_j + h'_{3j}M_j + \ell'_{3j}P_j \Big] \frac{\partial w}{\partial y} \\ & + \Big[g'_{1j}\left(\frac{\partial w}{\partial x}\right) + g'_{3j}\left(\frac{\partial w}{\partial y}\right) \Big] \varepsilon^L_j \\ & + \Big[h'_{1j}\left(\frac{\partial w}{\partial x}\right) + h'_{3j}\left(\frac{\partial w}{\partial y}\right) \Big] M_j \\ & + \Big[\ell'_{1j}\left(\frac{\partial w}{\partial x}\right) + \ell'_{3j}\left(\frac{\partial w}{\partial y}\right) \Big] P_j \Big\} \delta\left(\frac{\partial w}{\partial x}\right) \\ & + \Big\{ \Big(g'_{2j}\varepsilon^N_j + 2g'_{23}\varepsilon^N_1 + g'_{22}\varepsilon^N_2 \\ & + g'_{2j}\varepsilon^L_j + h'_{2j}M_j + \ell'_{j}P_j \Big) \frac{\partial w}{\partial y} \end{aligned}$$

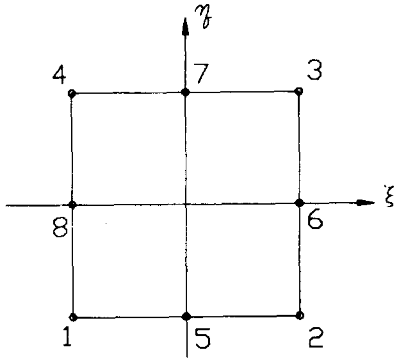

Fig. 1 Isoparametric quadratic element

$$+\Big[2g'_{3j}\varepsilon^N_j+\Big(\frac{1}{2}g'_{12}-g'_{33}\Big)\varepsilon^N_3$$
$$-g'_{3j}\varepsilon^L_j+h'_{3j}M_j+\ell'_{3j}P_j\Big]\frac{\partial w}{\partial x}$$
$$+\Big[g'_{3j}\Big(\frac{\partial w}{\partial x}\Big)+g'_{2j}\Big(\frac{\partial w}{\partial x}\Big)\Big]\varepsilon^L_j$$
$$+\Big[h'_{3j}\Big(\frac{\partial w}{\partial x}\Big)+h'_{2j}\Big(\frac{\partial w}{\partial y}\Big)\Big]M_j$$
$$+\Big[\ell'_{3j}\Big(\frac{\partial w}{\partial x}\Big)+\ell'_{2j}\Big(\frac{\partial w}{\partial x}\Big)\Big]P_j\Big\}\delta\Big(\frac{\partial w}{\partial y}\Big)$$
$$+\Big\{\Big[h'_{1j}\Big(\frac{\partial w}{\partial x}\Big)+h'_{3j}\Big(\frac{\partial w}{\partial y}\Big)\Big]\frac{\partial w}{\partial x}$$
$$+\Big[h'_{2j}\Big(\frac{\partial w}{\partial y}\Big)+h'_{3j}\Big(\frac{\partial w}{\partial x}\Big)\Big]\frac{\partial w}{\partial y}\Big\}\delta M_j$$
$$+\Big\{\Big[\ell'_{1j}\Big(\frac{\partial w}{\partial x}\Big)+\ell'_{3j}\Big(\frac{\partial w}{\partial y}\Big)\Big]\frac{\partial w}{\partial x}$$
$$+\Big[\ell'_{2j}\Big(\frac{\partial w}{\partial y}\Big)+\ell'_{3j}\Big(\frac{\partial w}{\partial x}\Big)\Big]\frac{\partial w}{\partial y}\Big\}\delta P_j\Big]dxdy \qquad (22)$$

In(21) and (22), $i,j=1,2,3$ and $m,n=1,2$ which are dummy indices representing Einstein's notation. $\boldsymbol{\varepsilon}^L_j,\boldsymbol{\varepsilon}^N_j,\boldsymbol{k}^0_j,\boldsymbol{k}^2_j,M_j,P_j$ represent the 3 components of $\boldsymbol{\varepsilon}^0{}_L,\boldsymbol{\varepsilon}^0{}_N,\boldsymbol{k}^0,\boldsymbol{k}^2,\boldsymbol{M},\boldsymbol{P}$ respectively and $\bar{\varepsilon}^0{}_m$ means the two components of $\bar{\varepsilon}^0$.

These linear and nonlinear stiffness matrices are calculated from (21) and (22), using 8-node isoparametric quadratic element as shown in Fig. 1.

$$x=\sum_{i=1}^{8}S_i(\xi,\eta)x_i$$
$$y=\sum_{i=1}^{8}S_i(\xi,\eta)y_i \qquad (23)$$

$$(u,v,w,\varphi_x,\varphi_y,M_x,M_y,M_{xy},P_x,P_y,P_{xy})$$
$$=\sum_{i=1}^{8}S_i(\xi,\eta)$$
$$(u_i,v_i,w_i,\varphi_{xi},\varphi_{yi},M_{xi},M_{yi},M_{xyi},P_{xi},P_{yi},P_{xyi}) \qquad (24)$$

S_i $(i=1,2,\cdots,8)$ included in the Eqs. (23), (24) are interpolation functions(Cook, 1989) of element nodes.

4. NUMERICAL RESULTS

Using the mixed type finite element method formulated in this paper, the linear and nonlinear behaviors of laminated composite plates are presented. According to the variations of plate thickness, layer number and fiber orientation, the effects on the deflections and the stresses of rectangular laminate plates are investigated. The finite element solutions are certified by comparing the results with the 3-dimensional exact solutions(ELASTIC) (Pagano, 1970), the higher order shear deformation theory (HSPT) (Reddy, 1984), the classical plate theory (CPT).

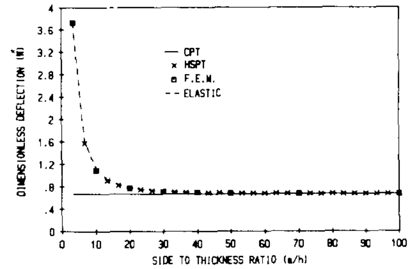

Fig. 2 Center deflection according to the variation of plate thickness : simply supported, $b/a=1$, (0/90/0)

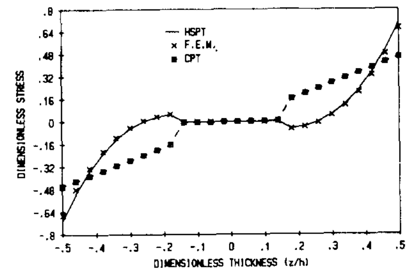

Fig. 3 Distribution of normal stress($\sigma^*{}_x$) through plate thickness : $b/a=1$, (0/90/0)

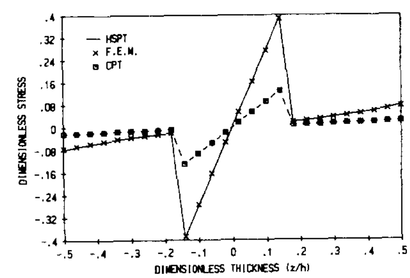

Fig. 4 Distribution of normal stress($\sigma^*{}_y$) through plate Thickness : $b/a=1$, (0/90/0)

Center deflections and interlaminar stress components of laminated square plates ($b/a=1$) subject to sinusoidal loading are shown in Figs. (2)～(7). These plates are assumed to be simply supported and have 3 layers of equal thickness, cross-ply, symmetric laminates (0/90/0) and equal material properties in each layer.

$$E_1/E_2=25,\quad G_{12}/E_2=0.5$$
$$G_{23}/E_2=0.2,G_{31}/E_2=0.5$$
$$\nu_{12}=0.25 \qquad (25)$$

Deflections and stresses were non-dimensionalized as follows ;

$$w^*=w\frac{E_2h^3}{q_0a^4}\times 100$$
$$(\sigma^*{}_x,\sigma^*{}_y,\tau^*{}_{xy})=\frac{h^2}{q_0a^2}(\sigma_x,\sigma_y,\tau_{xy}) \qquad (26)$$
$$(\tau^*{}_{yz},\tau^*{}_{zx})=\frac{h}{q_0a}(\tau_{xy},\tau_{zx})$$

In Fig. 2 the result of finite element method is in very good agreement with two analytical solutions, but if side to thickness ratio(a/h) is less than 50, CPT solution underestimates the deflection because of neglecting the shear effect on the transverse deformation. The variations of plane stress components through plate thickness are shown in Figs. 3, 4, and 5. CPT solution also underestimates the stresses and the differences are especially high on the layer boundaries.

Figures. 6 and 7 show the distributions of transverse shear stresses, which are underestimated by CPT. In the CPT solutions, these shear stresses are computed by integrating the following equilibrium equations and boundary conditions.

$$\frac{\partial \tau_{xz}}{\partial z} = -\frac{\partial \sigma_x}{\partial x} - \frac{\partial \tau_{xy}}{\partial y}$$

$$\frac{\partial \tau_{yz}}{\partial z} = -\frac{\partial \tau_{xy}}{\partial \sigma} - \frac{\partial \tau_y}{\partial y} \tag{27}$$

$$\tau_{xz}|_{z=\pm h/2} = 0, \quad \tau_{yz}|_{j=\pm h/2} = 0$$

Figure 8 shows the nonlinear results which the center deflection of orthogonalized square plate subjected to uniformly distributed loading is compared with the experimentally measured values (Zaghloul, 1975). This plate is simply supported. The results of finite element method are in good agreement within 5% but CPT solution is underestimated in nonlinear analysis too. This CPT solution (Zaghloul, 1975) is the result of finite difference nonlinear analysis for the same plate. Figures 9 and 10 show the variations of plate deflection

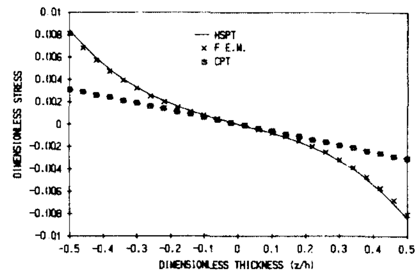

Fig. 5 Distribution of shear stress(τ^*_{xy}) through plate thickness: $b/a=1$, (0/90/0)

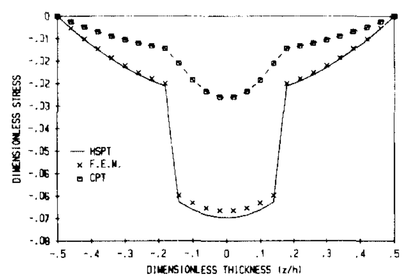

Fig. 6 Distribution of interlaminar shear stress(τ^*_{yz}) through plate thickness: $b/a=1$, (0/90/0)

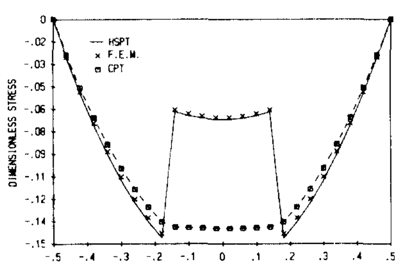

Fig. 7 Distribution of interlaminar shear stress(τ^*_{xz}) through plate thickness: $b/a=1$, (0/90/0)

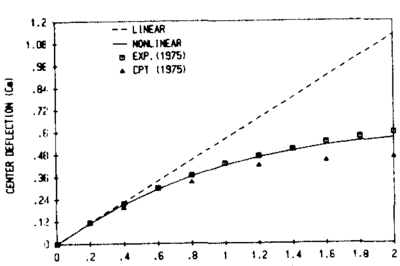

Fig. 8 Nonlinear deflection compared with experimentally measured values: simply supported, $b/a=1$

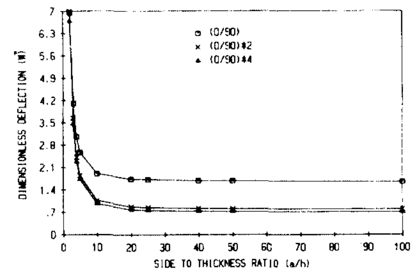

Fig. 9 Reduction of deflection according to increase of layer number in cross-ply laminate: (0/90)n

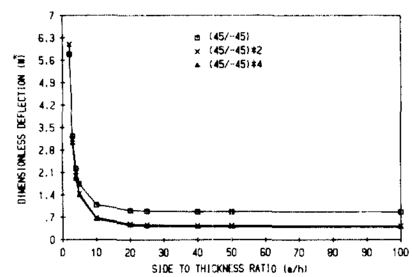

Fig. 10 Reduction of deflection according to increase of layer numner in angle-ply laminate: (45/-45)n

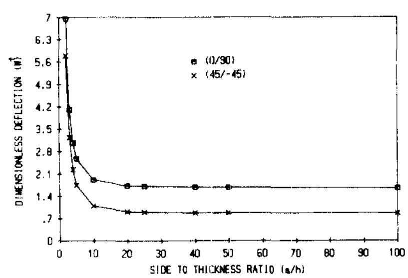

Fig. 11 Reduction of deflection due to fiber orientation: b/a=1, simply supported, uniformly distributed loading

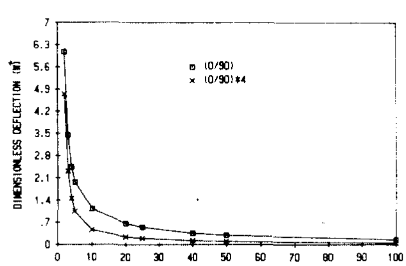

Fig. 12 Nonlinear effect on deflection according to layer numbers in cross-ply laminate: (0/90)n

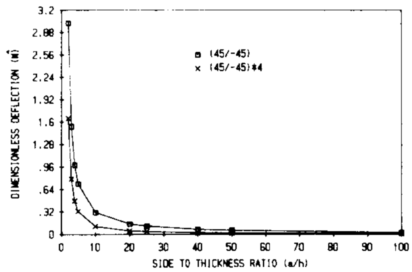

Fig. 13 Nonlinear effect on deflection according to layer number in angle-ply laminate : (45/-45)n

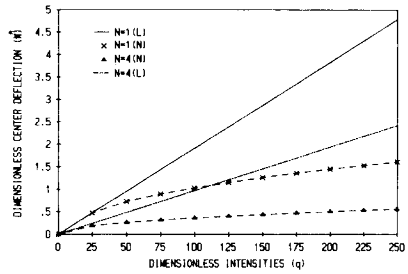

Fig. 14 Reduction of nonlinear effect according to layer number : (0/90)n, $b/a=1$, simply supported

according to layer numbers. In these figures, the deflection of a plate with the same thickness can be reduced by means of increasing layer numbers. These characteristics are almost constant without any relations of plate thickness and fiber orientation. For another method to reduce the deflection, as shown in Fig. 11, angle-ply (45/45)n prefers to the cross-ply (0/90)n in assembling the fiber orientation. Figs. 12 and 13 show that the nonlinear characteristics of the same plates are different from the linear ones. In a word, the reduction rates of deflection according to increasing layer numbers is more significant in thick plates.

This is considered as the reason why the nonlinear effect on the transverse shear deformation is reduced in the laminated plate with more layers. The nonlinear characteristics according to the variations of loadings are shown in Fig. 14. These nonlinear characteristics are increased according to the magnitude of loadings, but can be reduced by means of increasing layer numbers.

5. CONCLUSIONS

Higher order shear deformation theory of laminated composite plates is formulated by the mixed type finite element method. Laminated rectangular plates with the variations of thickness, layer number and fiber orientation are analyzed linearly and geometrically nonlinearly. As a result of this study, the following conclusions are obtained ;

(1) The mixed formulation of 8-noded isoparametric element with quadratic interpolation function is algebraically simpler than the general displacement type, and these results are in good agreement with the existing analytical solutions and the experimentally measured values.

(2) Deflection of a laminate plate with constant thickness can be reduced by means of increasing the layer numbers and assembling the fiber orientation in the angle-ply rather than the cross-ply.

(3) Increasing the layer numbers in the laminated composite plates has the effects of reducing the shear deformation and the nonlinear deflection.

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